Joinability is Undecidable for Ordered Rewriting

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In [CNNR03] several properties of ordered rewriting systems were investigated. An open question posed in the conclusion was, whether the joinability of two terms is decidable for ordered rewriting. In this note we show that this is not the case. It is undecidable. As corollaries we get that ground joinability, reachability, and normal form reachability are undecidable as well.

We use standard concepts from term rewriting [Ave95, BN98, DP01]. An *ordered rewrite system* is a triple $(R, E, \succ)$, where R is a finite set of rules, E is a finite set of equations and $\succ$ is a reduction ordering. We assume $R \subseteq \succ$ (i. e., $l \succ r$ for each $l \to r$ in R). With $E^{\succ}$ we denote the set of *orientable instances* of E, that is

$$E^{\succ} = \{\sigma(u) \to \sigma(v) \mid u \doteq v \text{ in } E,\ \sigma(u) \succ \sigma(v)\} .$$

We call rewriting with $R(E) = R \cup E^{\succ}$ *ordered rewriting*. Note that $R(E)$ is terminating by definition.

We consider the following decision problems:

Joinability. *Input:* Ordered rewrite system $(R, E, \succ)$, terms s and t. *Question:* Does $s \downarrow t$ hold (i. e., does there exist a term u such that $s \xrightarrow{*}_{R(E)} u \xleftarrow{*}_{R(E)} t$) ?

Ground joinability. *Input:* Ordered rewrite system $(R, E, \succ)$, terms s and t. *Question:* Does $s \Downarrow t$ hold (i. e., does $\sigma(s) \downarrow \sigma(t)$ hold for each ground substitution σ) ?

Reachability. *Input:* Ordered rewrite system $(R, E, \succ)$, terms s and t. *Question:* Does $s \xrightarrow{*}_{R(E)} t$ hold?

Normal form reachability. *Input:* Ordered rewrite system $(R, E, \succ)$, terms s and t, where t is in normal form. *Question:* Does $s \xrightarrow{*}_{R(E)} t$ hold?

In the proof we make use of the following decision problem [HU79], which we will use in a construction similar to [KNO90, Sect. 6]:

DEFINITION 1 *The* Modified Post Correspondence Problem *(MPCP) is the following. Given two finite lists $A = (u_1, \ldots, u_n)$ and $B = (v_1, \ldots, v_n)$ of words from Σ^+, does there exist a finite sequence $i_1, i_2, \ldots, i_k \in \{2, 3, \ldots, n-1\}$ such that*

$$u_1 u_{i_1} u_{i_2} \ldots u_{i_k} u_n = v_1 v_{i_1} v_{i_2} \ldots v_{i_k} v_n \ ?$$

The difference between MPCP and the usual PCP is that a solution is required to start with the first word on each list and end with the last word on each list. The MPCP is undecidable in general, because the halting problem of Turing machines can be reduced to this problem.

THEOREM 1 *It is undecidable in general, whether in a finite ordered rewrite system $(R, E, \succ)$ two ground terms s and t are joinable.*

PROOF By reduction of MPCP. Let $A = (u_1, \ldots, u_n)$ and $B = (v_1, \ldots, v_n)$ be an arbitrary instance of MPCP over alphabet $\Sigma = \{a_1, \ldots, a_m\}$. Let $sig = (\mathcal{S}, \mathcal{F}, \alpha)$ with sorts $\mathcal{S} = \{S\}$ and operators $\mathcal{F} = \{f, g_1, \ldots, g_n, h, a_1, \ldots, a_m, b, c\}$, where f has arity four, b and c are constants, and all other function symbols are unary. The function symbols $a_1, \ldots, a_m$ are used to encode a word $u \in \Sigma^+$ by the term $\hat{u}(x)$ in the usual way. For example, if $u \equiv a_1 a_2 a_2$, then $\hat{u}(x) \equiv a_1(a_2(a_2(x)))$. The general idea is to represent a potential solution $i_1, i_2, \ldots, i_k \in \{2, 3, \ldots, n-1\}$ of (A, B) by the ground term $f(\hat{u}_1 \hat{u}_{i_1} \hat{u}_{i_2} \cdots \hat{u}_{i_k} \hat{u}_n(b),\ g_1 g_{i_1} g_{i_2} \cdots g_{i_k} g_n(b),\ \hat{v}_1 \hat{v}_{i_1} \hat{v}_{i_2} \cdots \hat{v}_{i_k} \hat{v}_n(b),\ b)$. The function symbols g_i, $i = 1, \ldots, n$, encode the integer sequence of the solution.

Let $\succ$ be the LPO for $h >_{\mathcal{F}} f >_{\mathcal{F}} g_1 >_{\mathcal{F}} \cdots >_{\mathcal{F}} g_n >_{\mathcal{F}} a_1 >_{\mathcal{F}} \cdots >_{\mathcal{F}} a_m >_{\mathcal{F}} b >_{\mathcal{F}} c$. Note that $h(s) \succ t$ for all $s \in \mathrm{Term}(\mathcal{F}, \mathcal{V})$ and $t \in \mathrm{Term}(\mathcal{F} - \{h\})$. Especially, $h(c) \succ t$ iff $t \in \mathrm{Term}(\mathcal{F} - \{h\})$. The system $R \subseteq \succ$ consists of the following n rules:

$$\begin{aligned}
&f(\hat{u}_1(x), g_1(y), \hat{v}_1(z), b) \to f(x, y, z, c) \\
&f(\hat{u}_i(x), g_i(y), \hat{v}_i(z), c) \ \to f(x, y, z, c) \qquad \text{for } i = 2, \ldots, n-1 \\
&f(\hat{u}_n(b), g_n(b), \hat{v}_n(b), c) \to c
\end{aligned}$$

Because of the g_i, $i = 1, \ldots, n$, there exist no overlaps, hence R is confluent. It is easy to see that $\sigma(f(x, y, z, b)) \xrightarrow{*}_R c$ iff $\sigma(f(x, y, z, b))$ represents a solution to (A, B). The system E consists of equation $h(x') = f(x, y, z, b)$ only. Because of E, the ordered rewrite system $(R, E, \succ)$ is neither confluent nor ground confluent. For example, the term $h(b)$ rewrites with $E^{\succ}$ to the two irreducible terms $f(b, b, b, b)$ and $f(c, c, c, b)$.

Now consider the joinability in $(R, E, \succ)$ of the ground terms $s \equiv h(c)$ and $t \equiv c$. The term t is irreducible, for it is the smallest term with respect to $\succ$. The term s is R-irreducible, but ordered rewrite steps with $h(x') = f(x, y, z, b)$ are possible. As the equation contains extra variables, which are not determined by the match, we have $s \longrightarrow_{E^{\succ}} \sigma(f(x, y, z, b))$ for all substitutions σ such that $s \succ \sigma(f(x, y, z, b))$. This means that $\sigma(f(x, y, z, b)) \in \mathrm{Term}(\mathcal{F} - \{h\})$. Such terms are irreducible by $E^{\succ}$ and rewriting with R preserves this property. So the possible normal form derivations of s are of the form $s \longrightarrow_{E^{\succ}} \sigma(f(x,y,z,b)) \xrightarrow{!}_R s_\sigma$. As R is convergent the normal form s_σ is solely determined by σ. Especially, $s_\sigma \equiv c$ iff σ describes a solution to (A,B).

Hence, s and t are joinable in $(R,E,\succ)$ iff (A,B) has a solution. □

Note that in the previous construction terms s and t are ground and term $t \equiv c$ is in normal form. Hence the following corollary is immediate:

Corollary 1 *The decision problems Joinability, Ground joinability, Reachability, and Normal form reachability are undecidable.*

This theorem might be surprising as the rewrite relation $R(E)$ is terminating. Furthermore, ground confluence of an ordered rewrite system $(R,E,\succ)$ is decidable if $\succ$ is an LPO by the method described in [CNNR03]. The difference lies in the used ordered rewrite relation. For the confluence trees of [CNNR03] the relation $E_\succ$ is used, with

$$E_\succ = \{\sigma(u) \to \sigma(v) \mid u \doteq v \text{ in } E,\ \sigma(u) \succ \sigma(v),\ \text{and}\ \sigma(x) \equiv c_{\min} \text{ for all } x \in \mathrm{Var}(v) - \mathrm{Var}(u)\}\ ,$$

where $c_{\min}$ is the $>_{\mathcal{F}}$-smallest constant in $\mathcal{F}$. For Theorem 1, however, we make extensive use of extra variables and their various instances contained in $E^{\succ}$. For finite E, because of the extra variables, $E^{\succ}$ may be infinitely branching, whereas $E_\succ$ is always finitely branching. This indicates that it is far from trivial to replace $E^{\succ}$ by $E_\succ$ for testing ground confluence giving further support for the significance of the decidability result of [CNNR03].

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